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Casimir Force in Compact Noncommutative Extra Dimensions and Radius Stabilization

Soonkeon Nam[1]

Department of Physics
Harvard University
Cambridge, MA 02138
nam@pauli.harvard.edu

Abstract

We compute the one loop Casimir energy of an interacting scalar field in a compact noncommutative space of $R^{1,d} \times T^2_\theta$, where we have ordinary flat $1+d$ dimensional Minkowski space and two dimensional noncommuative torus. We find that next order correction due to the noncommutativity still contributes an attractive force and thus will have a quantum instability. However, the case of vector field in a periodic boundary condition gives repulsive force for $d > 5$ and we expect a stabilized radius. This suggests a stabilization mechanism for a senario in Kaluza-Klein theory, where some of the extra dimensions are noncommutative.

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[1] Permanent Address : Dept. of Physics, Kyung Hee University; Seoul, 130-701, Korea, nam@khu.ac.kr

An old idea, viewed in a new light, can have a new meaning. One of the oldest ideas of unifying gravity with other interactions is that of Kaluza[1] and Klein[2], i.e. that of extra dimensions. Its modern rebirth came about with the advent of supergravity theories, and it is an essential element in developments of string/M-theory.

More recently, new possibilities of large extra dimension[3] make the Kaluza-Klein modes something to be sought after in experiments [4]. For example, in some models[5] with more than two extra dimensions, there is a distinct mass gap in Kaluza-Klein spectrum[6][7]. So this eighty-years-old idea is still very much alive and is more so these days.

One question which is essential in Kaluza-Klein theories is how we can have small (or unobserved) extra dimensions. To explain smallness of extra dimensions, Appelquist and Chodos[8] suggested that the vacuum fluctuations of the higher dimensional gravitational field might provide a physical mechanism. They considered linearized quantum gravity in D-dimensions and computed the effective potential to one loop. For the effective potential, they obtained an infinite constant part[2], which is an induced cosmological constant and an attractive part. They computed the one loop vacuum energy in the compact extra dimensions, i.e. the gravitational Casimir energy[9]. Since the attractiveness of the Casimir energy pushed the size of the extra dimension down to the Planck scale, the natural cutoff scale of the linearized gravity, the *hope* was that presumably the dynamics of Planck scale, where the nonperturbative quantum gravity sets in, will stabilize the size of the extra dimensions. Thermal effects could not introduce any stabilty: either the size is pushed down to zero or to the infinity[10]. A very large number of light matter fields (around 10^{4-5}) can be introduced to stabilize the radius[11], since the gravitational contribution per degree of freedom to the Casimir energy is much larger than the matter contribution.

It is very natural to expect that there will be stabilization of the size of the extra dimensions if there is some intrinsic *minimum* length scale in the theory. One candidate certainly is the Planck (or string) scale as mentioned above. In this paper we will explore another possibility, when there is noncommutativity of space in the extra dimensions. When we have spacetime noncommutativity, Lorentz invariance is broken. However, having extra dimensions with broken (or deformed) Lorentz symmetry is not incompatible with observations so far.

There has been a lot of attention recently on quantum field theories on noncommutative spaces[12]. Interacting scalar field theories[13][14], QED[15] and other theories

[2] This infinity can of course be removed by proper dimensional regularization.

were[16] considered. This class of field theories is very interesting, because it arises naturally in the context of string theory[17], and is a consistent theory by itself. Here we have the following commutation relations among space-time coordinates x^μ

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{1}$$

and $\theta^{\mu\nu}$ introduces *minimum area* in the μ, ν plane, just as there is a minimum volume in phase space due to $[x, p] = i\hbar$, due to a space-time uncertainty relations

$$\Delta x^\mu \Delta x^\nu \geq \frac{1}{2}|\theta^{\mu\nu}|. \tag{2}$$

So in a noncommutative space there will be a length scale associated with $\sqrt{|\theta^{\mu\nu}|}$. Another consequence of this relation is the UV/IR mixing, due to the absence of decoupling of scales. Short distance scales in one direction is related to long distance scales in another direction which is related to the previous one by the parameters $\theta^{\mu\nu}$.

Difference with the Planck scale is that this scale is something which should be determined by underlying dynamics. In this sense we will not be able to solve the problem of radius stabilization in Kaluza-Klein theories completely. However, in most compactification senarios in string theory, $B_{\mu\nu}$ has expectation value[18], and we will be able to relate the expectation value of $B_{\mu\nu}$ with the radius of the Kaluza Klein radius.

This is interesting because, despite all the theoretical interests, the relevance of quantum field theories in noncommuatative space to measurable effects in particle physics has not been discussed very much. One of the main reason is that the presence of the external magnetic field which induces the noncommutativity breaks Lorentz invariance of the spacetime and thus a strong noncommutativity might not be a desirable thing to have. However, having a noncommutative extra dimension can be interesting, without destroying the desirable four dimensional Lorentz invariance. Very recently, there has been a work by Gomis et al[19] in this direction where the Kaluza-Klein spectrum due to noncommutative compact extra dimension has been considered. They obtain corrections to the Kaluza-Klein spectrum which resembles the contributions of winding states in string theory. This is consistent with the close relations found between string theory calculations and field theory calculations in noncommutative space [20][21].

In this paper, we explore possible effect that the noncommutativity in the extra dimensions might have on the Casimir force[22][23]. What we expect is that noncommutativity naturally introduces a minimum volume, that of the Moyal cell, which is proportional to the noncommutativity parameter and will eventually stabilize the radius. Therefore it is quite natural to expect that it will compete with the attractive Casimir force. In this paper, we find an interesting result that actually depending on the type of the field, scalar or vector, the next order correction due to the noncommutativity will be either attractive or repulsive. The theory with scalar field with phi^3 interaction has quantum instability whereas a theory with vector field stabilizes.

To be more quantitative, let us begin by a simple review of the derivation of the Casimir effect, when a massless scalar field is confined between two parallel plates separated by a distance a[9][24]. (Similar analysis can certainly be done for massive fields, with similar but more complicated results[24].) Although Dirichlet, Neumann, or in general Robin (i.e. mixed)[25]boundary condition can be dealt with, we consider the Dirichlet case for simplicity.

$$\phi(0) = \phi(a) = 0. \tag{3}$$

The Casimir force between the plates is obtained by summing up the zero-point energy per unit area,

$$E/A = \frac{1}{2}\sum_{\alpha}\omega_\alpha = \frac{1}{2}\sum_{n=1}^{\infty}\int\frac{d^2k}{(2\pi)^2}\sqrt{k^2+\frac{n^2\pi^2}{a^2}}, \tag{4}$$

where for convenience we have set $\hbar = c = 1$. The integers $n = 1, 2, \cdots$ label the normal modes between the plates and k is the transverse momenta along the plates. The sum as it stands is formally divergent.

To extract a finite value from this divergent sum, we invoke dimensional regularization. Using the definition of the Gamma function,

$$\Gamma(z) = p^z \int_0^\infty e^{-pt}t^{z-1}dt, \tag{5}$$

we can put the energy per unit volume A (at the boundary) E/A as follows:

$$E/A = \frac{1}{2}\mu^{3-d}\sum_{n=1}^{\infty}\int\frac{d^dk}{(2\pi)^d}\int_0^\infty\frac{dt}{t}t^{-1/2}e^{-t(k^2+n^2\pi^2/a^2)}\frac{1}{\Gamma(-\frac{1}{2})}. \tag{6}$$

In the above we have introduced an arbitrary mass scale μ to keep above expression a four dimensional energy density. From now on we will suppress the dependences on μ. Now we interchange the integrations over t and k and integrate over the transverse momenta, which is the Gaussian integration. The final result is

$$E/A = -\frac{1}{a^{d+1}}\Gamma\left(\frac{d+2}{2}\right)(4\pi)^{-(d+2)/2}\zeta(d+2). \tag{7}$$

In obtaining this equation we have used the following reflection formula of Riemann zeta function to avoid infinities from $\Gamma(z)$ when z is a negative even integer:

$$\Gamma\left(\frac{z}{2}\right)\pi^{-z/2}\zeta(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{(z-1)/2}\zeta(1-z). \tag{8}$$

Since the energy is always is negative and falls off as a decreases, we always have attractive force between the plates due to the massless scalar field.

Let us now generalize this to the noncommutative ϕ^3 field theory on $R^{1,d} \times T^2_\theta$. By this we mean that

$$[x^{d+1}, x^{d+2}] = i\theta, \tag{9}$$

and other commutation relations among space time coordinates vanish. Also $0 \leq x^{d+1}, x^{d+2} \leq 2\pi R$. We must have an interacting theory to see the effects of the noncommutativity. In a perturbative quantum field theory in noncommutative space, all the information about the noncommutativity can be put in the interaction vertices of Feynman diagrams. Moreover, the effect of noncommutativity appears through nonplanar diagrams.

The theory we are considering is defined by the following action[13]:

$$S = \int d^{d+3}x \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi\star\phi\star\phi\right). \tag{10}$$

The $\star$-product is defined by

$$(f\star g)(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\alpha^\mu}\frac{\partial}{\partial\beta^\mu}} f(x+\alpha)g(x+\beta)|_{\alpha=\beta=0}, \tag{11}$$

and introduces an infinite number of higher derivative interactions. The kinetic part is not modified with the noncommutativity.

One can calculate the one loop contributions to the two point functions and obtain the one loop corrections to the dispersion relation for nonzero modes in the Kaluza-Klein spectrum[19], when we have periodic boundary condition. The Kaluza-Klein spectrum at one loop is as follows:

$$m^2_{\vec{n}} = m^2 + \frac{\vec{n}^2}{R^2} - \frac{\lambda^2}{(4\pi)^3}\left(\frac{R^2}{\vec{n}^2\theta^2} + \frac{5}{24}m^2\ln\left(\frac{m^2\theta^2\vec{n}^2}{R^2}\right) + \cdots\right). \tag{12}$$

In the above, $\vec{n} = (n_{d+1}, n_{d+2})$ are integers which give the quantized momenta $\vec{p} = \vec{n}/R$, along the compact directions. From now on we will simply put $\vec{n} = (n_1, n_2)$. This mass formula resembles that of winding states in string theory. The mass correction is negative, and for small values of θ the mass spectrum becomes tachyonic. This certainly is not a healthy sign. Either there is an intrinsic instability in the theory or the perturbative analysis is not adequate for small θ's. In this paper, we will stay in the region where the perturbative analysis is valid. We will only consider the case of massless scalar field, for the sake of simplicity.

Casimir energy is again obtained by summing up all the modes, to obtain the energy density in $d+1$ dimensions as follows:

$$u = E/A = \frac{1}{2}\sum_{\alpha}\omega_\alpha = \frac{1}{2}\sum_{n_1,n_2=1}^{\infty}\int \frac{d^d\vec{k}}{(2\pi)^3}\sqrt{k^2+\frac{\vec{n}^2}{R^2}-\frac{\lambda^2 R^2}{(4\pi)^3\theta^2\vec{n}^2}}. \tag{13}$$

For the square root we can introduce the Schwinger's proper time representation:

$$u = \frac{1}{2}\sum_{n_1,n_2}\int\frac{d^dk}{(2\pi)^d}\int_0^\infty \frac{dt}{t}t^{-1/2}e^{-t\left(\vec{k}^2+\vec{n}^2/R^2-\lambda^2R^2/(4\pi)^3\theta^2\vec{n}^2\right)}. \tag{14}$$

Now we integrate the transverse momentum k, by doing a Gaussian integral,

$$u = \frac{-1}{4\sqrt{\pi}}\frac{1}{(4\pi)^{d/2}}\sum_{n_1,n_2}\int_0^\infty\frac{dt}{t}t^{-(d+1)/2}e^{-t\left(\vec{n}^2/R^2-\lambda^2R^2/(4\pi)^3\theta^2\vec{n}^2\right)}. \tag{15}$$

Again we perform the t integration using the integral representation of Gamma function and obtain

$$u = \frac{-1}{4\sqrt{\pi}}\frac{1}{(4\pi)^{d/2}}\Gamma\left(-\frac{d+1}{2}\right)\sum_{n_1,n_2}\left(\frac{\vec{n}^2}{R^2}-\frac{\lambda^2R^2}{(4\pi)^3\theta^2\vec{n}^2}\right)^{(d+1)/2}. \tag{16}$$

The infinite sum as it stands does not admit a closed form. However, if we consider only the perturbative regime, then we can expand in terms of $\lambda^2a^2/\theta^2 \ll 1$, and have the following:

$$\begin{aligned} u &= \frac{-1}{4\sqrt{\pi}}\frac{1}{(4\pi)^{d/2}}\Gamma\left(-\frac{d+1}{2}\right)\sum_{n_1,n_2}\left(\frac{\vec{n}^2}{R^2}\right)^{(d+1)/2}\left[1-\frac{d+1}{2}\frac{\lambda^2R^4}{(4\pi)^3\theta^2(\vec{n}^2)^2}+\cdots\right], \\ &= \frac{-1}{2R^{d+1}}\frac{1}{(4\pi)^{(d+1)/2}}\Gamma\left(-\frac{d+1}{2}\right)\left[v_2\left(\frac{-d-1}{2}\right)-\frac{d+1}{2}\frac{\lambda^2R^4}{(4\pi)^3\theta^2}v_2\left(\frac{-d+3}{2}\right)+\cdots\right]. \end{aligned} \tag{17}$$

In the above we have used the following notation and a relation found by Hardy[26]:

$$v_2(s) \equiv \sum_{m,n=1}^{\infty}(m^2+n^2)^{-s} = \zeta(s)\beta(s)-\zeta(2s), \tag{18}$$

where $\beta(z)$ is defined as follows[27]:

$$\beta(z) = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-z}. \tag{19}$$

The value of this function is between 1/2 and 1 for real positive z's and oscillates with unbound amplitude for negative real values of z. Note that the Gamma function in the above function is divergent for odd d's. We can use the 'reflection formula'[28]:

$$\Gamma(z)\zeta(z)\beta(z)\pi^{-z} = \Gamma(1-z)\zeta(1-z)\beta(1-z)\pi^{z-1}. \tag{20}$$

So the structure of the energy per unit area is

$$u = -\alpha/R^{d+1} - \gamma/R^{d-3}. \tag{21}$$

where both α and γ are both positive constants. α is the coefficients one gets in the commutative case, and the ratio γ/α is

$$\frac{\gamma}{\alpha} = \frac{-1}{d-1} \frac{\pi\lambda^2}{8\theta^2} \frac{\Gamma\left(\frac{d-1}{2}\right)\zeta\left(\frac{d-1}{2}\right)\beta\left(\frac{d-1}{2}\right) - \Gamma\left(\frac{d-2}{2}\right)\zeta(d-2)\sqrt{\pi}}{\Gamma\left(\frac{d+3}{2}\right)\zeta\left(\frac{d+3}{2}\right)\beta\left(\frac{d+3}{2}\right) - \Gamma\left(\frac{d+1}{2}\right)\zeta(d+1)\sqrt{\pi}}. \tag{22}$$

We see that the contribution from the noncommutative part will never become repulsive and will not stabilize the size of radis R even for $d > 3$. The case of $d = 3$ the contribution from the correction to the Casimir energy constant, so there is no repulsive force. So in the case of $d = 3$, we see that up to the order of perturbation we have used, there is no stabilization. So we might say that we have to consider the next order in correction to the Casimir energy, i.e. higher order terms in Eq. 18. Of course, in order to be really consistent we first need the result for two loop self energy and it is beyond the scope of this paper.

In order to discuss the higher dimensional tori, each direction having different radius, we can perform a similar calculation. Again the vacuum energy is

$$\frac{1}{2}\sum_{n=1}^{\infty}\int \frac{d^d k}{(2\pi)^3}\sqrt{\vec{k}^2 + \sum_i \left(\frac{n_i{}^2}{R_i^2} - \rho^2 \frac{R_i^2}{n_i^2\theta^2} + \cdots\right)}. \tag{23}$$

In the above we have indicated the next order correction on dimensional ground the next order correction.

For this we need Epstein Zeta function[29]

$$Z_p(1/a_1,\cdots,1/a_p;s) = \sum_{n_1=-\infty}^{\infty}\cdots\sum_{n_p=-\infty}^{\infty}{}' \left[\left(\frac{n_1}{a_1}\right)^2+\cdots+\left(\frac{n_p}{a_p}\right)^2\right]^{-s/2}. \qquad (24)$$

Here the prime denotes that the term for which all $n_i = 0$ is to be omitted. The qualitative feature will be similar, in the sense that there will a attractive contribution, in the noncommutative limit, and the noncommutative part will have attractive (and sometimes marginal) contribution. This can be seen in the limit where the effects of the nonplanar part is maximal. Of course for a quantitative result we have to resort to numerial methods. We have seen that the noncommutative extra dimensions cannot be stabilized with scalar fields with the introduction of the noncommutativity.

First of all this result should be generalized for the case of vector and linearized gravity. Since in a noncommutative spacetime a pure $U(1)$ gauge theory is *interacting* unlike in ordinary space, we will be able to see the effects of noncommutativity in this theory. Consider the action

$$S = -\frac{1}{4}\int d^{d+3}xF_{MN}\star F^{MN}, \qquad (25)$$

where the field strength is

$$F_{MN} = \partial_M A_N - \partial_N A_M - ig(A_M\star A_N - A_N\star A_M), \qquad (26)$$

Actually the Kaluza-Klein spectrum from a noncommutative $U(1)$ gauge theory in six dimensions is available[19], and is given by the following formula:

$$m_{\vec{n}}^2 = \frac{\vec{n}^2}{R^2} - \frac{8g^2R^4}{\pi^3\theta^4(\vec{n}^2)^2} + \cdots. \qquad (27)$$

There is a similarity with the scalar field case, but also a difference. The dependence on the radius of the extra dimension R is different and has a higher power, and this will affect the vacuum energy.

Following a similar analysis we have

$$\begin{aligned} u &= \frac{-2}{4\sqrt{\pi}}\frac{1}{(4\pi)^{d/2}}\Gamma\left(-\frac{d+1}{2}\right)\sum_{n_1,n_2}\left(\frac{\vec{n}^2}{R^2}\right)^{(d+1)/2}\left[1-\frac{d+1}{2}\frac{8g^2R^6}{\pi^3\theta^4(\vec{n}^2)^3}+\cdots\right], \\ &= \frac{-1}{R^{d+1}}\frac{1}{(4\pi)^{(d+1)/2}}\Gamma\left(-\frac{d+1}{2}\right)\left[v_2\left(\frac{-d-1}{2}\right)-\frac{d+1}{2}\frac{8g^2R^6}{\pi^3\theta^4}v_2\left(\frac{-d+5}{2}\right)+\cdots\right]. \end{aligned} \qquad (28)$$

We have multiplied the polarization factor of 2. A rigorous proof that the case of electromagnetic field fluctuations give the same result as the scalar case with a factor of two can be given[23]. Since the ratio $\frac{v_2((-d+5)/2)}{v_2((-d-1)/2)}$ stays positive for all values of $d > 5$ we now expect that the Casimir force will stabilize for $d > 5$ and not for $d \leq 4$. The value of the ratio decreases with increasing d. The compactification radius will be at

$$R = R_0 = \left(\left(\frac{2}{d-5} \right) \left(\frac{\pi^3 \theta^4}{8g^2} \right) \left(\frac{v_2\left(\frac{-d+5}{2}\right)}{v_2\left(\frac{-d-1}{2}\right)} \right) \right)^{\frac{1}{6}}, \tag{29}$$

when there is only a $U(1)$ vector field present.

This is to be contrasted to what we had for the scalar field. It is expected that similar Kaluza-Klein spectrum for gravity will lead to similar results. Since the Casimir force for the gravity is far more dominant than that of scalar fields, the field which is responsible for the compactifiction will be the gravity field and we expect to have a stabilization of the size of the extra dimensions. Of course, it would be interesting to investigate these cases in detail.

Recently, Casimir force between branes in a flat S^1/Z_2 orbifold compactification of M-theory was computed[30]. Here one finds similar dependence in the distance between the branes, as in the case with circle compactification[31]. It is expected that for the cases with more extra dimensions, similar analysis as performed here will give a 'stabilization' in the distances between the branes. Another interesting work is by Goldberger and Rothstein[32] on the quantum stabilization of radion stabilization[33]. The system is such that it consists of two branes bounding a region of anti de Sitter space. It turns out that the quantum fluctuation destabilizes the system, just as in the case of flat space. It would be interesting to consider the consequences of noncommutativity in the context of radion stabilization and study the quantum stability, which we do expect.

The Casimir effect in supergravity theories in a supersymmetric backgrounds, have cancellation of the contribution from bosonic part by the fermionic part[34]. However, a finite temperature breaks supersymmetry and there will be a finite Casimir effect in a senario of early universe.

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